Exercise 46

(a) Find the asymptotes of the graph of $f(x) = \dfrac{4-x}{3+x}$ and use them to sketch the graph.

(b) Use your graph from part (a) to sketch the graph of f'

(c) Use the definition of a derivative to find $f'(x)$

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

Solution

Part (a)

To determine the vertical asymptote(s), set what's in the denominator equal to zero and solve for x.

$$3 + x = 0$$

$$x = -3$$

To determine the horizontal asymptote(s), evaluate the limit of $f(x)$ as $x \to \pm\infty$ and set the result(s) equal to y.

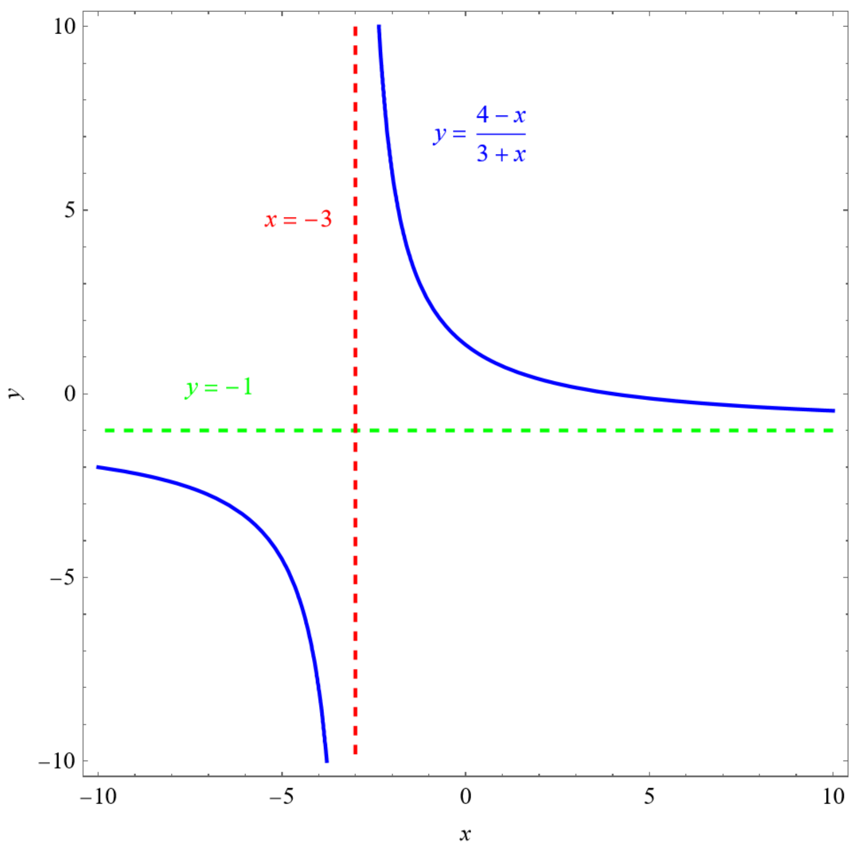
$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4-x}{3+x} \\
&= \lim_{x\to\infty} \frac{4-x}{3+x} \times \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{4}{x}-1}{\frac{3}{x}+1} \\
&= \frac{0-1}{0+1} \\
&= -1
\end{aligned}$$

One horizontal asymptote is therefore $y = -1$.

In the limit as $x \to -\infty$, make the substitution $u = -x$ so that as $x \to -\infty$, $u \to \infty$.

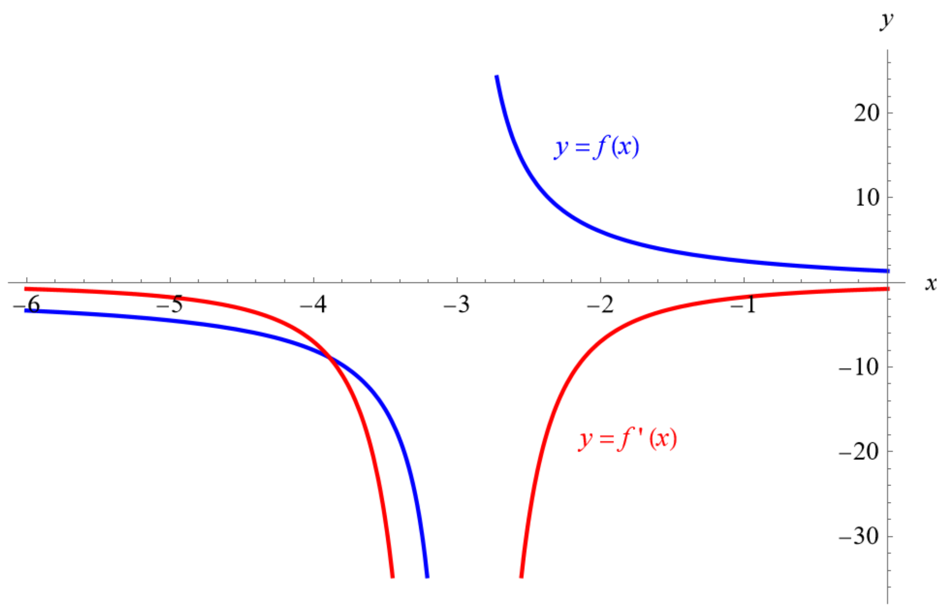
$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{4-x}{3+x} \\ &= \lim_{u\to\infty} \frac{4-(-u)}{3+(-u)} \\ &= \lim_{u\to\infty} \frac{4+u}{3-u} \\ &= \lim_{u\to\infty} \frac{4+u}{3-u} \times \frac{\frac{1}{u}}{\frac{1}{u}} \\ &= \lim_{u\to\infty} \frac{\frac{4}{u}+1}{\frac{3}{u}-1} \\ &= \frac{0+1}{0-1} \\ &= -1\end{aligned}$$

$y = -1$ is the only horizontal asymptote. Below is a graph of $f(x)$ together with its asymptotes.



Part (b)

Below is a graph of $f(x)$ and $f'(x)$ versus x.



Part (c)

Use the definition of a derivative to calculate $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{4-(x+h)}{3+(x+h)} - \frac{4-x}{3+x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{4-x-h}{3+x+h} - \frac{4-x}{3+x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(3+x)(4-x-h)}{(3+x)(3+x+h)} - \frac{(4-x)(3+x+h)}{(3+x)(3+x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(3+x)(4-x-h)-(4-x)(3+x+h)}{(3+x)(3+x+h)}}{h} \\
&= \lim_{h\to 0} \frac{(3+x)(4-x-h)-(4-x)(3+x+h)}{h(3+x)(3+x+h)} \\
&= \lim_{h\to 0} \frac{(12-3h+x-x^2-xh)-(12+x+4h-x^2-xh)}{h(3+x)(3+x+h)}
\end{aligned}$$

Simplify the numerator, cancel h, and evaluate the limit.

$$f'(x) = \lim_{h\to 0} \frac{-7h}{h(3+x)(3+x+h)}$$

$$= \lim_{h\to 0} \frac{-7}{(3+x)(3+x+h)}$$

$$= \frac{-7}{(3+x)(3+x)}$$

$$= -\frac{7}{(3+x)^2}$$

Part (d)

Below is a graph of $f'(x)$ versus x.